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Statistical dynamics of two-dimensional flow

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The equilibrium statistical mechanics of inviscid two-dimensional flow are re-examined both for a continuum truncated at a top wavenumber and for a system of discrete vortices. In both cases, there are negative-temperature equilibria for finite flows. But for spatially infinite flows, there are only positive-temperature equilibria, and both the continuum and discrete system exhibit proper, extensive, thermodynamic limits at all realizable values of the energy and enstrophy density. The negative-temperature behaviours of the continuum and discrete system are semi-quantitatively the same, except for a supercondensation phenomenon in the discrete case at the smallest realizable values of negative temperature. The supercondensed states have very large energy and in them all vortex cores of the same sign are clumped within an area small compared with the mean area per vortex. The approach of the continuum system to absolute equilibrium by enstrophy cascade to high wavenumbers and energy cascade to low wavenumbers is examined. It is argued that the enstrophy cascade is closely analogous to distortion of a passive scalar field by straining of large spatial scale. This implies that high intermittency of spatial derivatives of the vorticity field can develop but that there is no associated change in the previously proposed log-corrected k^{-1} enstrophy spectrum law. On the other hand, intermittency build-up in the downward energy cascade can result in a change of the exponent in the energy spectrum law to a negative value of smaller magnitude than $\frac{5}{3}$. Intermittency effects in the non-equilibrium energy cascade seem a more plausible explanation for vortex clumping observed in recent computer experiments than do the spatially smooth condensation phenomena associated with the negative-temperature absolute equilibria.

1. Introduction

The statistical mechanics of two-dimensional flow were first discussed by Onsager (1949), who treated an assemblage of discrete point vortices by means of a Hamiltonian formalism. Onsager found that absolute equilibrium ensembles of high kinetic energy exhibited negative temperatures and corresponded to clumping of like-signed vortices. The equilibrium statistical mechanics of two-dimensional flow with a continuous vorticity distribution were treated by Kraichnan (1967). Again, the absolute equilibrium states of high relative kinetic energy corresponded to negative temperatures. The equilibrium statistics were found to be closely analogous to those of a perfect boson gas, with the particle number and kinetic energy in the boson gas playing the respective roles of kinetic energy and enstrophy (integrated squared vorticity) in the fluid. Both systems exhibited condensed states with singularly large excitation of the modes of lowest wavenumber.

In inviscid two-dimensional flow, the vorticity of each fluid element is a constant of motion. One consequence is that, in addition to kinetic energy, enstrophy is an inviscid constant of motion. This has profound effects on non-equilibrium as well as equilibrium statistics. Fjørtoft (1953) pointed out that, in contrast to three-dimensional flow, the two constants of motion implied that any transfer of energy to higher wavenumbers must be accompanied by a bigger transfer to lower wavenumbers. Elaboration of these considerations led Batchelor (1969), Kraichnan (1967) and Leith (1968) to propose a dual inertial-cascade mechanism for two-dimensional turbulence. In this picture, energy cascaded from input wavenumbers to lower wavenumbers through an inertial range with energy spectrum

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}} \tag{1.1}$$

and enstrophy cascaded to higher wavenumbers through an inertial range of the form

$$E(k) = C'\eta^{\frac{2}{3}}\,k^{-3}, \tag{1.2}$$

where C and C' are dimensionless constants, ϵ is the rate of energy cascade and η is the rate of enstrophy cascade, both per unit mass. Corrections for non-local interactions in the wavenumber space modify (1.2) to the form (Kraichnan 1971; Leith & Kraichnan 1972)

$$E(k) = C'\eta^{\frac{2}{3}}k^{-3}[\ln(k/k_1)]^{-\frac{1}{3}} \quad (k \gg k_1), \tag{1.3}$$

where k_1 is characteristic of the input wavenumbers.

In the past few years there has been a renewed interest in two-dimensional turbulence, in part because it is now possible to do direct computer simulations (Deem & Zabusky 1971; Lilly 1971, 1972*a*, *b*; Fox & Orszag 1973; Herring *et al.* 1974) and in part because of analogies with and applications to meteorological flows (Charney & Stern 1962) and plasma dynamics. The relation to plasma physics lies in the fact that the dynamical equations for an assemblage of point vortices are identical with those for charge filaments in the so-called guiding-centre plasma. The latter is a plasma in which charge filaments aligned parallel to a uniform magnetic field move perpendicular to the field under their mutual electric field. Detailed investigations of the equilibrium statistical mechanics of the guiding-centre plasma, starting from Onsager's formulation and giving special emphasis to the negative-temperature states, have been reported by Joyce & Montgomery (1973), Edwards & Taylor (1974) and others.

The present paper has two principal purposes. The first is to present in some detail the equilibrium statistical mechanics of continuum two-dimensional flow and point out the correspondence between continuum results and those for discrete-vortex systems. A particular objective is to resolve some confusion about the nature and significance of the negative-temperature states which we feel has appeared in the literature. The second main purpose is to examine the effect on the proposed inertial-range spectra (1.1) and (1.3) of intermittency phenomena of the sort which, in three dimensions, are widely thought to modify the Kolmogorov inertial-range spectrum.

Kolmogorov (1962) and Oboukhov (1962) suggested that spatial intermittency which increased in a self-similar fashion with decreasing scale size should alter the exponent in (1.1) to the form $-\frac{5}{3}-\mu$, where $\mu > 0$, in three dimensions. Since then there has been a substantial amount of experimental and qualitative theoretical support for this hypothesis. A critical review of the theoretical arguments was attempted by Kraichnan (1974*a*). This raises the question of whether similar effects should make the exponent in (1.2) or (1.3) more negative in two-dimensional enstrophy cascade and whether there is a corresponding modification of (1.1) in two dimensions.

In the present paper, we attack these questions by considering the energy and enstrophy cascades in the context of the broader problem of the relaxation of the two-dimensional flow system towards absolute statistical equilibrium. We shall also make use of analogies between enstrophy cascade and the distortion of a convected passive scalar field. As part of the investigation of intermittency effects, we shall examine the suggestion by Saffman (1971) that intermittency of the small scales takes an extreme form in which the major contribution to the mean-square vorticity gradient comes from boundary layers between large-scale eddies. This would make $E(k)$ proportional to k^{-4} instead of taking the form (1.3). A central question in the overall problem of relaxation towards absolute statistical equilibrium is the nature of the constraints imposed by the existence of an infinity of local inviscid constants of motion: the vorticity of each fluid element.

2. The absolute equilibrium distributions

If the velocity field in a cyclic box of period D is expanded in a Fourier series of the form

$$\tilde{u}_i(\mathbf{x},t) = \sum_{\mathbf{k}} u_i(\mathbf{k},t)\exp(i\mathbf{k}.\mathbf{x}), \tag{2.1}$$

the incompressible Navier–Stokes equation takes the form

$$(\partial/\partial t+\nu k^2)\,u_i(\mathbf{k}) = -ik_m P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p})\,u_m(\mathbf{q}). \tag{2.2}$$

Here the sums are over all wave vectors allowed by the cyclic boundary conditions, ν is the kinematic viscosity and $P_{ij}(\mathbf{k})$ is the transverse projection operator which embodies the pressure forces.

The total energy (divided by density) and total enstrophy in the cyclic box are

$$D^2\sum_{\mathbf{k}}|\mathbf{u}(\mathbf{k})|^2 \equiv D^2\hat{E}$$

and

$$D^2\sum_{\mathbf{k}}k^2|\mathbf{u}(\mathbf{k})|^2 \equiv D^2\hat{\Omega}$$

respectively. Here enstrophy is defined as half the integrated squared vorticity and $|\mathbf{u}(\mathbf{k})|^2$ denotes $u_i(\mathbf{k})\,u_i^*(\mathbf{k})$. If $\nu = 0$, both energy and enstrophy are conserved and, moreover, they are conserved individually by each triad of interacting wave vectors $\{\pm\mathbf{k}, \pm\mathbf{p}, \pm\mathbf{q}\}$ (cf. Kraichnan 1973, §3). The detailed conservation properties imply that energy and enstrophy are inviscidly conserved if the dynamical system is truncated by removing from all the sums every wave vector $\mathbf{k}, \mathbf{p}$ or $\mathbf{q}$ whose magnitude does not lie between a lower cut-off k_0 and an upper cut-off $k_{\max}$.

A further property when $\nu = 0$ is that the motion of phase points, and of hyperplane projections of this motion, is divergence free if a phase space is constructed whose Cartesian co-ordinates are real and imaginary parts of the vector components of the Fourier amplitudes (Lee 1952). Thus,

$$\partial \dot{a}_i(k)/\partial a_i(\mathbf{k}) + \partial \dot{b}_i(\mathbf{k})/\partial b_i(\mathbf{k}) = 0, \tag{2.3}$$

where $u_i(\mathbf{k}) = a_i(\mathbf{k}) + ib_i(\mathbf{k})$.† This detailed Liouville theorem continues to hold if the inviscid system is truncated at the limits k_0 and $k_{\max}$. It then follows that any probability density function in the phase space which is a function $P(\hat{E}, \hat{\Omega})$ of only the constants of motion $\hat{E}$ and $\hat{\Omega}$ is itself time-invariant. This result is an immediate consequence of (2.3), the continuity equation

$$\partial P/\partial t + \sum_{\mathbf{k}} \partial/\partial u_i(\mathbf{k})\,[\dot{u}_i(\mathbf{k})\,\partial P/\partial u_i(\mathbf{k})] = 0, \tag{2.4}$$

and the inviscid identities†

$$\sum_{\mathbf{k}} \dot{u}_i(\mathbf{k})\,\partial \hat{E}/\partial u_i(\mathbf{k}) \equiv 0, \quad \sum_{\mathbf{k}} \dot{u}_i(\mathbf{k})\,\partial \hat{\Omega}/\partial u_i(\mathbf{k}) \equiv 0, \tag{2.5}$$

which express the fact that $\hat{E}$ and $\hat{\Omega}$ are constants of motion.

Two distributions of special importance are (in unnormalized form)

$$P = \exp(-\alpha\hat{E} - \beta\hat{\Omega}) \tag{2.6}$$

and

$$P = \delta(\hat{E} - E)\,\delta(\hat{\Omega} - \Omega). \tag{2.7}$$

Here (2.6) generalizes the canonical distribution of ordinary Hamiltonian mechanics. Also, it is homologous to the grand canonical ensemble of a free-boson gas in the classical-field limit, as we shall discuss later. The special property of (2.6) is that it is stable under arbitrary intercouplings of the systems of the ensemble, provided that these couplings preserve the form and constancy of both $\hat{E}$ and $\hat{\Omega}$ (Kraichnan 1959). Thus (2.6) is a thermal equilibrium ensemble, with α playing the role of inverse temperature and β acting as a corresponding thermodynamic potential (which we shall resist naming) for enstrophy. Alternatively, in the boson analogy, α/β and β are the chemical potential and inverse temperature, respectively. The distribution (2.7) is appropriate to an isolated system of energy E and enstrophy Ω, under the assumption that it exhibits suitable ergodic properties. We shall first discuss (2.6).

Equation (2.6) yields the mean modal intensity spectrum

$$U(k) = \tfrac{1}{2}(\beta k^2 + \alpha)^{-1}, \tag{2.8}$$

where

$$U(k) \equiv (D/2\pi)^2 \langle |\mathbf{u}(\mathbf{k})|^2 \rangle \tag{2.9}$$

† The reality condition $u_i^*(\mathbf{k}) = u_i(-\mathbf{k})$ requires that, in the partial differentiations of (2.3)–(2.5), the independent variables be taken as either $a_i(\mathbf{k})$ and $b_i(\mathbf{k})$ for $\mathbf{k}$ in a half-space or $u_i(\mathbf{k})$ for all $\mathbf{k}$.

and $\langle\ \rangle$ denotes a normalized average over P. The usual isotropic energy spectrum is given by $E(k) = \pi k U(k)$, provided that k_0 is high enough that the modes are dense (mode spacing small compared with k) over the entire spectrum. In that case, the mean energy $E = \langle \hat{E} \rangle$ and mean enstrophy $\Omega = \langle \hat{\Omega} \rangle$ obtained by integrating over (2.8) are

$$E = (\pi/4\beta)\ln[(\alpha+\beta k_{\max}^2)/(\alpha+\beta k_0^2)], \tag{2.10}$$

$$\Omega = (\pi/4\beta)(k_{\max}^2 - k_0^2) - (\pi\alpha/4\beta^2)\ln[(\alpha+\beta k_{\max}^2)/(\alpha+\beta k_0^2)]. \tag{2.11}$$

These thermal equilibrium formulae exhibit three regimes, distinguished by the signs of α and β and by the value of $k_1^2 \equiv \Omega/E$ (Kraichnan 1967). Let

$$k_a^2 = \tfrac{1}{2}(k_{\max}^2 - k_0^2)/\ln(k_{\max}/k_0), \quad k_b^2 = \tfrac{1}{2}(k_{\max}^2 + k_0^2).$$

The regimes are then

$$\text{(I)}\ \ k_0^2 < k_1^2 < k_a^2, \qquad \beta > 0, \quad -\beta k_0^2 < \alpha < 0; \tag{2.12}$$

$$\text{(II)}\ \ k_a^2 < k_1^2 < k_b^2, \qquad \alpha > 0, \quad \beta > 0; \tag{2.13}$$

$$\text{(III)}\ \ k_b^2 < k_1^2 < k_{\max}^2, \qquad \alpha > 0, \quad -\alpha < \beta k_{\max}^2 < 0. \tag{2.14}$$

Regime II is the most ordinary one. In it, $U(k)$ decreases monotonically with increasing k. It is bounded by the energy-equipartition state $\beta = 0$, $k_1 = k_b$ and the enstrophy equipartition state $\alpha = 0$, $k_1 = k_a$. Regimes I and III feature negative values of α and β, respectively. They are more unusual and are the ones which have led to perplexity and confusion in the literature. Cook & Taylor (1972) and others have suggested that the negative-temperature states of regime I are not true equilibrium states to which systems can relax. In answer to this, we have already noted that all three regimes are not only equilibria in the sense that (2.6) is time invariant but that these equilibria are *stable* under arbitrary couplings that preserve the constants of motion. The important thing here is that there are two constants of motion. Neither positive-temperature nor negative-temperature equilibria survive coupling to a reservoir in such a way that both constants are not conserved. If the reservoir conserves only energy, then the only distribution of the form (2.6) which can survive coupling is the energy-equipartition state $\beta = 0$.

Fox & Orszag (1973) have pointed out that there is no discontinuous change of property of any kind in crossing the boundaries between the regimes. This follows immediately from the fact that (2.6) is normalizable and an analytic function of α and β over all three regimes. Regimes I and III are most interesting in the limits $k_1 \to k_0$ and $k_1 \to k_{\max}$ and in the limits $k_0 \to 0$ and $k_{\max} \to \infty$ for fixed k_1. If $k_1 - k_0 \ll k_0$, then $\alpha \approx -\beta k_0^2$ and $U(k)$ exhibits a sharp peak at $k = k_0$. As the limit $k_1 = k_0$ is approached, the fraction of the total energy contained in this peak approaches unity. Complementary behaviour occurs at the opposite limit. If $k_{\max} - k_1 \ll k_{\max}$, then $\beta k_{\max}^2 \approx -\alpha$ and $U(k)$ exhibits a sharp peak at $k_{\max}$. As the limit $k_1 = k_{\max}$ is approached, the fraction of the total enstrophy contained in this peak approaches unity.

Now let k_1 be fixed while $k_{\max} \to \infty$ with given k_0. It follows readily from (2.10) and (2.11) that

$$\beta \to \pi k_{\max}^2/2\Omega, \quad \alpha/\beta + k_0^2 \to k_{\max}^2 \exp(-k_{\max}^2/k_1^2). \tag{2.15}$$

As the limit is approached, α becomes negative whatever the values of k_0 and k_1. On the other hand, if $k_0 \to 0$ with fixed k_1 and k_{max}, α and β are both positive as the limit is approached. If, in addition, $k_{max} \gg k_1$, then (2.15) is valid in this case also. Thus negative-temperature equilibrium states occur only when the lowest admitted wavenumber k_0 is non-zero.

We have so far dealt only with dense mode distributions so that k in (2.8) can be treated as a continuous variable. This implicitly assumes that the cyclic box size $D \to \infty$ before $k_0 \to 0$. A more physically relevant situation is to take $k_0 = 2\pi/D$. In this case the low-lying modes cannot be considered dense, and the structure of the negative-α states is affected. As $k_1 \to k_0$ at fixed k_0 and k_{max}, the excitation in the modes of wavenumber k_0 increases such that, as the limit is approached, these modes carry most of the kinetic energy. In the limit, only these modes are excited. As $D \to \infty$ ($k_0 \to 0$) at fixed k_1 and k_{max}, again α and β both are positive as the limit is approached. The negative-temperature states exist only for a finite fluid and are absent for an infinite one.

In visualizing the negative-α states, it is important to realize that the condensation of kinetic energy affects only modes of wavenumber k_0 in the case where $k_0 = 2\pi/D$ and $k_1 - k_0 \ll k_0$. For $k = 2k_0$, the denominator of (2.8) already is insignificantly different from what it would be with $\alpha = 0$. If $k_1 - k_0 \sim k_0$, the energy condensation is unimportant even at k_0. Thus the condensed states do not involve an intermittent spatial distribution of big vortices placed or moving at random in the fluid. They differ from positive-energy states only in the spatially smooth excitation of the lowest modes to high levels. The stream function for this excitation has the general form

$$\psi = a\cos[k_0(x-x_0)] + b\cos[k_0(y-y_0)], \tag{2.16}$$

where a and b are coefficients, x and y are co-ordinates in the plane of the flow and x_0 and y_0 define an origin which may be anywhere in the cyclic cell. If $a = b$, this motion consists of a single pair of counter-rotating vortices per cell. Note that uniform translational motion $k = 0$ is dynamically uncoupled from the modes $k > 0$ in the sense that $k = 0$ motion is never excited if it is not present initially. We therefore exclude it completely.

The equilibrium distributions are essentially unchanged if the cyclic boundaries are replaced by rigid, no-slip boundaries and the square is replaced by an arbitrary boundary shape. The general case can be formulated by expanding the velocity field in eigenfunctions of $\nabla^4\phi = \lambda^4\phi$. For rectangular boundaries with cyclic or slip conditions, this reduces to a Fourier expansion, as in (2.1). For rectangular boundaries with no-slip conditions, the appropriate expansion involves both Fourier modes and the modified Fourier modes introduced by Chandrasekhar & Reid (1957). Thus,

$$\tilde{u}_x(x,y) = \sum a_{nm} C_n(x)\, c_m(y), \quad \tilde{u}_y(x,y) = \sum b_{nm} c_n(x)\, C_m(y), \tag{2.17}$$

where the C_n are Chandrasekhar–Reid functions and the c_n are ordinary Fourier functions. Both kinetic energy and enstrophy are a sum of squares of the a and b coefficients and the equilibrium law takes a form closely analogous to (2.8), with similar behaviour in the several limiting cases considered above. In contrast to the case of cyclic boundary conditions, the equilibrium distributions are, of course, not statistically homogeneous.

The generalized microcanonical distribution (2.7) can give averages appreciably different from those of (2.6) only when there is significant excitation in regions of k space where the modes are not dense; this means the negative-α distributions with $k_1 \sim k_0$. Even in this case, however, the differences do not seem crucial. The distribution (2.6) can be obtained by averaging (2.7) over values of E and Ω. When the resultant values of α and β yield $k_1^2 - k_0^2 \ll k_0^2$, then $\Omega/E \approx k_1^2$ for almost all the distributions (2.6) contributing to this average, since $\Omega/E < k_0^2$ is impossible. Thus (2.7) exhibits the same condensation of kinetic energy into k_0 as (2.6), when $k_1 \rightarrow k_0$. In the discrete-mode (finite box) case where $k_0 = 2\pi/D$, both distributions yield a singular concentration of kinetic energy into k_0, in the limit $k_1 \rightarrow k_0$, while the excitation of the remaining modes exhibits near equipartition of enstrophy.

The expressions for energy and enstrophy given after (2.2) are identical in form with those for particle number and kinetic energy, respectively, in the classical-field limit of a quantized boson field, where each degree of freedom is typically excited by many quanta. Consequently there is an exact correspondence between the equilibrium statistics discussed above and those of the boson field, provided that the latter is truncated at wavenumbers k_0 and $k_{\max}$. The parameter β in the expressions above represents inverse temperature in the boson problem, while α/β is the chemical potential. In particular, the condensation of kinetic energy into k_0 discussed above corresponds to the Einstein–Bose condensation of particles into the ground state in a two-dimensional free-boson gas of finite size. In three dimensions, but not in two, the Einstein–Bose condensation occurs in the infinite gas also.

There are, however, two important differences between the two-dimensional inviscid Navier–Stokes fluid and the two-dimensional free-boson gas. In the Navier–Stokes case, the degrees of freedom are coupled (nonlinearly) by the equations of motion even though the energy is a simple sum of squares. Consequently, the Navier–Stokes fluid can relax towards equilibrium, while the perfect boson gas can do so only with the introduction of extra couplings. The second difference is that in a real Navier–Stokes fluid the artificial cut-off at $k_{\max}$ is replaced by the dissipative effects of viscosity, while in the boson case the effective cut-off is provided by non-dissipative quantum effects. Viscosity destroys the absolute equilibria while the quantum effects only modify the form of the equilibria.

3. Relation between continuous and discrete vorticity

Onsager (1949) initiated the statistical-mechanical study of the system of N isolated vortices in an inviscid two-dimensional flow. If the vortices are point vortices, their kinetic energy of interaction may be written in the Hamiltonian form

$$H = -(2\pi)^{-1}\rho \sum_{i>j} q_i q_j \ln (r_{ij}/r_0), \tag{3.1}$$

where q_i is the circulation about the ith vortex, r_{ij} is the distance between the ith and jth vortices, ρ is the density of the fluid (mass/area) and r_0 is an arbitrary length which gives a zero level to H. The Hamiltonian equations are

$$d\bar{x}_i/dt = \partial H/\partial \bar{y}_i, \quad d\bar{y}_i/dt = -\partial H/\partial \bar{x}_i, \tag{3.2}$$

where $\bar{x}_i = (\rho q_i)^{\frac{1}{2}} x_i, \bar{y}_i = (\rho q_i)^{\frac{1}{2}} y_i$ and (x_i, y_i) are the Cartesian co-ordinates of the ith vortex.

Equation (3.1) is the Hamiltonian for an infinite fluid. For fluid in a cyclic box of side D, the appropriate form is

$$H = \rho D^{-2} \sum_{i>j} \sum_{\mathbf{k}} q_i q_j k^{-2} \exp(i\mathbf{k}.\mathbf{r}_{ij}), \tag{3.3}$$

where all allowed $\mathbf{k}$ $(k > 0)$ are included. If the vorticity field $\tilde{\omega}(\mathbf{x})$, which here has the form

$$\tilde{\omega}(\mathbf{x}) = \sum_i \delta(\mathbf{x}-\mathbf{r}_i) q_i, \tag{3.4}$$

is expanded in the form

$$\tilde{\omega}(\mathbf{x}) = \sum_{\mathbf{k}} \omega(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x}), \tag{3.5}$$

then

$$\omega(\mathbf{k}) = D^{-2} \sum_i q_i \exp(-i\mathbf{k}.\mathbf{r}_i) \tag{3.6}$$

is a collective co-ordinate for the vortex system ($\mathbf{r}_i$ is the position (x_i, y_i) of the ith vortex). The Hamiltonian may be rewritten as

$$H = \tfrac{1}{2}\rho D^2 \sum_{\mathbf{k}} k^{-2}[|\omega(\mathbf{k})|^2 - \sum_i q_i^2], \tag{3.7}$$

where the sum over i represents the subtraction of the self-energy of the vortices and arises from the exclusion of $i = j$ in (3.3).

The amplitude $\omega(\mathbf{k})$, which is a pseudoscalar in two dimensions, is related to $u_i(\mathbf{k})$ by

$$\omega(\mathbf{k}) = i\epsilon_{ij} k_i u_j(\mathbf{k}), \quad u_i(\mathbf{k}) = i\epsilon_{ij} k_j k^{-2} \omega(\mathbf{k}), \tag{3.8}$$

where ϵ_{ij} is the alternating matrix $(\epsilon_{11} = \epsilon_{22} = 0, \ \epsilon_{12} = -\epsilon_{21} = 1)$. Hence the energy expression for the continuous vorticity distribution, given after (2.2), may be rewritten in precisely the form (3.7), *but without the self-energy subtraction.* For the continuum, the total kinetic energy is the natural quantity and self-energy is not a well-defined concept. For both the continuum and discrete system, the total energy is formally given by

$$\mathscr{E} = \tfrac{1}{2}\rho D^2 \sum_{\mathbf{k}} k^{-2} |\omega(\mathbf{k})|^2. \tag{3.9}$$

Physical intuition suggests that both the equilibrium and non-equilibrium behaviour of the continuum system should closely approximate that of the discrete system provided that two related conditions are fulfilled: first, that the continuum is truncated at $k_{\max} \sim n^{\frac{1}{2}}$, where $n = N/D^2$ is the discrete-system number density, and second, that close encounters (vortices much closer together than the mean spacing $n^{-\frac{1}{2}}$) do not play a significant role. Under these conditions, both systems have approximately the same number of degrees of freedom, and the singularly intermittent vorticity distribution of the discrete system plays no significant part; smearing of the vortex cores over a diameter $n^{-\frac{1}{2}}$ would not appreciably affect the $\omega(\mathbf{k})$ for $k < k_{\max}$ in the discrete system. A further implicit condition, since we are considering the reflexion-invariant ensembles with zero mean vorticity, is that all the $|q_i|$ are equal, with equal numbers of q_i with plus and minus signs.

The presumption of similar behaviour stated above can be precisely stated, without actually settling the real role played by close encounters, by truncating the discrete-system equations (3.3) and (3.7) at $k_{\max} \sim n^{\frac{1}{2}}$ also, thereby creating a Hamiltonian system with rigid vortex cores of effective widths $\sim n^{-\frac{1}{2}}$. We then anticipate that the continuum absolute equilibrium results of § 2, with $k_0 = 2\pi/D$, are accurate qualitative descriptions of the discrete-vortex equilibrium behaviour. Moreover, the correspondence should extend to more general k_0 values if a low wavenumber cut-off at k_0 is made in (3.3) and (3.7).

To formulate the correspondence in detail, we note that the typical value of $\tilde{\omega}(\mathbf{x})$ in the truncated discrete system is $qn = qk^2_{\max}$, so that $\Omega \sim q^2n^2$. For any state of the continuum system where most of the enstrophy is in near equipartition [almost all states and, in particular, the limits discussed in connexion with (2.15)], $\beta \sim k^2_{\max}/\Omega$, as in (2.15). Thus we make the choice $\beta \sim 1/(q^2n)$ to pick the appropriate continuum states. Moreover, the α of § 2 is related to the temperature T by $\alpha = \rho/k_B T$, where k_B is Boltzmann's constant. With these relations noted, we anticipate that the dependence on temperature of the mean energy $E = \langle \mathscr{E} \rangle/\rho D^2$ per unit mass is well approximated for the discrete system by (2.10).

Particular important results which carry over (with $k_0 = 2\pi/D$) are (i) the existence of negative-T equilibria and associated condensation phenomena for finite D; (ii) the disappearance of the negative-T states for infinite D; (iii) the existence of a well-behaved, extensive, thermodynamic limit in which, as $D \to \infty$, T approaches a finite positive limiting value for any given energy density E (provided that E sufficiently exceeds the minimum possible value $\Omega/k^2_{\max}$).

Edwards & Taylor (1974) have concluded, on the contrary, that the discrete-vortex system does not yield extensive thermodynamic limits with $k_0 = 2\pi/D$, whether or not it is truncated at $k_{\max}$. They ascribe this to the long-range nature of the 'Coulomb potential' which appears in (3.3). We believe that this conclusion comes from a physically unjustifiable use of the self-energy subtraction.

To see the physics most clearly, put the Fourier representation aside and regularize the singularities at short distances by supposing that each vortex of strength q has a core of radius a over which its vorticity is uniform. Consider an instantaneous state in which the vortices are placed completely at random in the volume D^2, with number density n. Outside the core, the velocity field of each vortex falls off like $|\mathbf{x}-\mathbf{r}_i|^{-1}$, so that the self-energy of each vortex is proportional to ln (D/a). Consequently, the total self-energy of all the vortices is $\sim \rho q^2 N \ln(D/a)$ and is not proportional to D as $D \to \infty$ at fixed n. On the other hand, the physical observable is the total kinetic energy, which is the integrated square of the actual velocity field throughout the volume. This energy *is* proportional to D as D increases at fixed n. The far velocity field induced by a local group of vortices falls off faster than $1/x$ because of cancellations between plus and minus circulations. Since the total energy is proportional to D^2 [it is $\sim \rho q^2 n D^2 \ln(1/n^{\frac{1}{2}}a)$ for the random state in question] while the self-energy is proportional to $D^2 \ln(D/a)$, the interaction energy must be negative and proportional to $D^2 \ln(D/a)$ as $D \to \infty$ at fixed n. If one examined only the interaction energy, one would falsely conclude that this state is not properly extensive.

The general thermal equilibrium state is more structured than the purely random configuration just discussed, but there is no escape from the same basic physics unless positive and negative vortices are highly segregated. But that corresponds to the negative-T states which disappear as $D \to \infty$. It is simply inadmissible to subtract off the self-energy, as is done in forming (3.3), if dependence on D is to be examined. The self-energy per vortex depends on D so its subtraction is *not* a trivial resetting of the energy zero level.

It is, however, possible to subtract off consistently the infinite energy associated with point vortices ($a \to 0$) while still retaining the essential dependence on D. This can be done, for example, by adding to (3.3) the modified self-energy term

$$H^*_{\text{self}} = \tfrac{1}{2}\rho D^{-2} \sum_i \sum_{k=k_0}^{k^*} q_i^2 k^{-2}, \tag{3.10}$$

where k^* is an arbitrary cut-off for which a natural choice would be $k^* = n^{\frac{1}{2}}$. The energy $H + H^*_{\text{self}}$ exhibits proper extensive behaviour.

In the context of the guiding-centre plasma, it should perhaps be emphasized that we are *not* talking about the self-energy of individual electrons. They are three-dimensional and their self-energy can safely be subtracted because it is independent of D as $D \to \infty$. The self-energy of present concern is, from the microscopic point of view, the *interaction* energy of the necessarily large number of electrons which make up a macroscopic charge filament of sufficient diameter and density that charge fluctuations within it are negligible.

In support of some of the preceding arguments, we wish to point out that the continuum results do, in fact, agree with an approximate treatment of the discrete system by Taylor (1972), in which the self-energy is retained. Taylor transforms the configuration integral in the relevant structure function into an integral over the collective co-ordinates $\omega(\mathbf{k})$, truncated at $k_{\max} \sim n^{\frac{1}{2}}$. This leads to an explicit relation between T and E. To see the correspondence with Taylor's result, we may rewrite (2.10) as

$$\alpha = \beta[k_{\max}^2 - k_0^2 \exp(4\beta E/\pi)]/[\exp(4\beta E/\pi) - 1]. \tag{3.11}$$

Taking $\beta \sim 1/(q^2 n)$ and noting $\alpha = \rho/k_B T$ as before, we find that (3.11) is identical with the final result [equation (6)] of Taylor's paper, apart from normalization conventions and factors of 2 and π which we have ignored in β.

In a later and more detailed paper, Edwards & Taylor (1974) drop the self-energy. This gives cancellations at large k which permit them to get finite results with $k_{\max} = \infty$. Here, as we would anticipate from the preceding, they do not find a proper thermodynamic limit for $D \to \infty$ with $k_0 = 2\pi/D$. Although this result comes about solely from dropping the self-energy, it should be pointed out that the limit $k_{\max} \to \infty$ is inadmissible in their treatment. The approximate transformation to collective co-ordinates as new variables, which is the key to their evaluation of the relation between T and E, can be justified only if $k_{\max} \sim n^{\frac{1}{2}}$, so that the new variables are approximately as numerous as the old.

It seems not unfair to say that any treatment which appeals to a truncated set of collective variables is a treatment of the discrete-vortex system in name only. Once the $\omega(\mathbf{k})$ for $k > k_{\max}$ are discarded, the analysis does not retain sufficient information to distinguish concentrated vortices from a continuous vorticity distribution. In fact, the continuum analysis with $k_{\max} \sim n^{\frac{1}{2}}$ has just as much claim to represent the discrete-vortex system as does the truncated collective-variable analysis, and it has the advantage of leading immediately to very simple, exact results.

We must now return to the matter of possible close encounters of the discrete vortices. They offer the only remaining mechanism of qualitatively different behaviour of the continuum and discrete system. It is clear from the preceding paragraph that analysis using truncated collective co-ordinates is no help here. However, considerable insight is provided by simple qualitative estimates of some *a priori* configuration probabilities and their associated energies in the canonical distribution.

Consider point vortices and let N and D have large fixed values. Take first the possibility of collapse of the system into neutral vortex pairs (Hauge & Hemmer 1971). Simple combinatorics show that the *a priori* probability of all configurations in which each vortex is a member of a neutral pair at separation not exceeding a distance $a \leqslant n^{-\frac{1}{2}}$ is $\sim (a^2 n)^{\frac{1}{2}N}$, where we neglect factors of strength e^{-N} and weaker. The total interaction energy for such a configuration is

$$H \sim -\rho q^2 N \ln (D/a).$$

Hence the product of the *a priori* probability and the factor $\exp(-H/k_B T)$ is

$$\sim N^{\frac{1}{2}\alpha q^2 N} (a^2 n)^{\frac{1}{2}(1-\alpha q^2)N},$$

where $\alpha = \rho/k_B T$ and we use $N/D^2 = n$. The condition that this expression should not diverge as $a \to 0$ is clearly $\alpha q^2 < 1$ or

$$k_B T > \rho q^2, \tag{3.12}$$

if $T > 0$. Thus collapse into neutral pairs does not occur if T is negative or is positive and sufficiently large.

This result has an interesting interpretation in terms of the correspondence between discrete and continuum systems. Taking $k_{\max}^2 \sim n$ and $\beta \sim 1/q^2 n$ as before, we find that $\alpha q^2 = 1$ corresponds to $\alpha \sim \beta k_{\max}^2$ for the continuum. If α takes larger positive values, then (2.8) goes over from enstrophy equipartition at the wavenumbers $k \sim k_{\max}$ to energy equipartition at these wavenumbers. Thus (3.12) says that the energy-equipartition states $\alpha > \beta k_{\max}^2$ of the continuum are not accessible to the discrete system, because of pair collapse.

If, instead, we construct the *a priori* probability and Boltzmann factor for close approach of like-signed pairs, the only change is in the sign of the interaction energy. Collapse now occurs if $\alpha < -q^2$ or

$$0 > k_B T > -\rho q^2. \tag{3.13}$$

However, this condition appears to be empty because macroscopic condensation phenomena place stronger constraints on the negative-T regime. If, again, $\beta \sim 1/q^2 n$ and $n = k^2_{\max}$ are used, $\alpha > -q^{-2}$ implies $0 > \alpha > -\beta k^2_{\max}$ for the negative-α continuum regime. But that regime already is constrained by the much stronger condition (2.12), with $\alpha = -\beta k_0^2$ corresponding to complete condensation into the mode k_0.

Our previous discussion shows that, if near encounters are excluded, the discrete system will show macroscopic condensation similar to that of the continuum for $\alpha \approx -\beta k_0^2$. Without this exclusion, there is, in addition, the possibility of a more extreme behaviour which we may call supercondensation. This is a state in which all vortex cores of the same sign are within an indefinitely small distance a of each other. The *a priori* probability for such a state is

$$\sim (a^2/D^2)^N = (a^2 n/N)^N,$$

while the interaction energy is $\sim \rho N^2 q^2 \ln(D/a)$. Thus the product of the *a priori* probability and Boltzmann factor can be estimated as

$$(N/a^2 n)^{-N-\frac{1}{2}\alpha N^2 q^2}.$$

The condition that this should not diverge as $a \to 0$ is

$$\alpha > -2q^{-2}/N, \tag{3.14}$$

a result which clearly makes empty the condition $\alpha > -q^{-2}$ for stability to like-signed pair collapse.

If $k^2_{\max} \sim n$, $\beta \sim 1/q^2 n$ and $k_0 = 2\pi/D$, then the most negative value of α permitted by (3.14) is $\sim -\beta k_0^2$, the maximum negative value found in §2 for the continuum. It is not possible, without a more precise treatment of the configuration statistics, to decide whether the discrete-vortex system first exhibits the continuum-type condensation or the more extreme supercondensation as α takes increasingly negative values (starting from zero) with fixed D, N and q. One possibility is the following. The continuum actually shows two stages of condensation as $\alpha \to -\beta k_0$. In the first, $k_1^2 \sim k_0^2$, $\alpha + \beta k_0^2$ is exponentially small, according to (2.15), but most of the enstrophy still lies in the higher k where there is approximate enstrophy equipartition. In the second stage, k_1^2 approaches k_0^2 so closely that most of the enstrophy, as well as most of the energy, lies in the peak at k_0. It seems possible that the discrete system may also show a two-stage behaviour in which, first, a condensation like that of the continuum occurs, with macroscopic segregation of different-signed vortices. Here the velocity field averaged over regions containing many vortices is concentrated in the continuum modes about k_0, but the typical vortex spacing is still $\sim n^{-\frac{1}{2}}$, giving a vorticity distribution that fluctuates on the scale $n^{-\frac{1}{2}}$ not too differently from the way it does in the uncondensed states. The second stage would be supercondensation, with a radically different vorticity fluctuation character and a total energy which displays an indefinite logarithmic increase as the supervortices become more concentrated.

Whether or not the supercondensation is preceded by a stage of continuum-type condensation, its onset plausibly could consist of the formation of local supervortices, made of relatively few individuals, which could then coalesce to form supervortices of greater intensity, until the final stage of total coalescence is reached.

A final point is that that supercondensation is an open possibility even for the distributed-core rigid-vortex system obtained by truncating (3.3) at $k_{\text{max}} \sim n^{\frac{1}{2}}$. In contrast to vortices in an actual fluid, which never overlap (in the absence of viscosity) if they do not initially, the vortices of this artificial Hamiltonian system can coincide.

4. Relaxation towards equilibrium

The inviscid equilibria of § 2 are both exact and stable, but that does not necessarily imply that they are the final states of evolution from particular initial conditions. The equilibria of § 2 might be forbidden because of additional constants of motion which we have not yet taken into account. There is also the question of the relevance of the finite-k_{max} results to the evolution of inviscid systems with k_{max} infinite.

When k_{max} is infinite, the vorticity of each fluid element is an inviscid constant of motion. This provides immediate constraints on the evolution of initial statistical distributions since it follows that the univariate vorticity distribution in $\mathbf{x}$ space is invariant. We shall make the hypothesis, in what follows, that the effect of detailed vorticity invariance on evolution is to produce a fine-graining of the vorticity field closely analogous to that of a passive scalar field convected by turbulence (Kraichnan 1974*b*). In the absence of molecular diffusion, the scalar field amplitude is constant in every fluid element. But either a non-zero diffusivity or a cut-off at finite k_{max} smears out the fine-graining of the scalar amplitude when the latter reaches small enough spatial scales. Detailed constancy of amplitude is lost and the remaining isolating constants of motion are the integrals, over space, of the scalar amplitude and its square (the latter only if the smearing is produced by finite k_{max} rather than finite diffusivity).

If the analogy is valid (we shall give supporting arguments but no proof), then detailed conservation of vorticity should not interfere with relaxation towards the equilibria of § 2 except for fine-graining of the spatial distribution of vorticity. If the vorticity distribution is averaged over small distances, the relaxation of systems with infinite k_{max} and with large finite k_{max} should be equivalent.

Consider, now, an initial state in which the velocity distribution is multivariate normal, with all excitation confined to a fairly narrow band about k_1. Assume $k_0 = 2\pi/D$ and let k_1 be sufficiently larger than k_0 that modes $k \gtrsim k_1$ are dense and the initial distribution can be very close to isotropic. If $k_{\text{max}} \gg k_1$, then the distribution of form (2.8) to which this initial state can relax is specified by (2.15).

It is easy to see, from (2.8), (2.10) and (2.15), that, even with $k_1 \gg k_0$, all of the energy in the equilibrium distribution lies in the peak at k_0 if $k_{max} \to \infty$. The enstrophy, on the other hand, is nearly in equipartition, with most of it in modes $k \sim k_{max}$, because that is where most of the modes are.

It should be stressed that even though all the energy is asymptotically in k_0, the equilibrium distribution does *not* exhibit a high degree of clumping of like-signed vorticity. This is evident from the fact that the total enstrophy is Ek_1^2, while that associated with k_0 is Ek_0^2, which is much smaller according to our assumption $k_1 \gg k_0$. Moreover, the initial state cannot evolve to a state exhibiting a high degree of such clumping, whether or not it evolves to our equilibrium. This follows directly from the conservation laws. The largest fraction of the initial enstrophy which can appear in like-signed vorticity clumps of scale k, without increasing the initial energy, is $\sim (k/k_1)^2$. Clearly the states considered by Onsager (1949), in which the vorticity field is dominated by a few large eddies, cannot be reached from our initial state. This kind of limitation, was, in fact, explicitly recognized by Onsager in his paper.

The fact that fluid elements with positively and negatively signed vorticity must be essentially uniformly distributed on spatial scales $\gtrsim k_1^{-1}$ favours a picture in which enstrophy is transferred to high wavenumbers by a straining process analogous to the straining of small blobs of passive scalar by turbulence (Batchelor 1959; Kraichnan 1974*b*). It works against the contrary picture proposed by Saffman (1971), in which the fine-scale vorticity structure is principally associated with boundary layers between macro-eddies. No substantial fraction of the original vorticity can be concentrated into boundary layers occupying a small fraction of the fluid. Nor, since vorticity and straining are intimately related, can straining be especially strong in such boundary layers. Thus it is hard to see how the proposed boundary-layer mechanism can compete effectively with straining of small-scale vorticity fluctuations throughout the fluid volume.

If the straining of small-scale vorticity fluctuations were exactly like that of a passive scalar, then the approach to equilibrium at high wavenumbers would involve enstrophy transfer through a range in which the enstrophy spectrum went as k^{-1}, like the analogous scalar spectrum, so that the energy spectrum $E(k)$ went like k^{-3} (Batchelor 1969; Leith 1968; Kraichnan 1967). We have just seen that this analogy is not flawed by segregation of like-signed vorticity. However, there are two further defects. The scalar k^{-1} range applies when the wavenumbers of the straining field are small compared with those of the strained scalar blobs. But in a k^{-1} enstrophy range, each octave in wavenumber makes an equal contribution to the straining, assuming similarity of the statistical distributions. This implies logarithmic corrections to the k^{-1} law (Kraichnan 1971; Leith & Kraichnan 1972). A more fundamental problem is that the vorticity field is not passive. It is functionally related to the velocity field and reacts upon it. We wish now to argue that this reaction has a negligible effect on the straining process in the asymptotic enstrophy cascade range.

If a k^{-1}-type enstrophy transfer range does develop, it carries enstrophy to higher k at a rate asymptotically independent of k as $k \to \infty$. If $k_{max} = \infty$, we anticipate that the k^{-1}-type range would extend to ever-increasing k as time increases, so that the equilibrium enstrophy-equipartition state would never be achieved. On the other hand, for finite $k_{\max}$ the enstrophy cascade should pump enstrophy to wavenumbers $\sim k_{\max}$ and provide a mechanism whereby equilibrium eventually is achieved. After considering, in the next section, the reaction of small-scale vorticity on the velocity field, we shall return to the equilibrium-approach problem, taking up intermittency effects both in the enstrophy cascade and in the complementary cascade of energy from $k \sim k_1$ down to k_0.

5. Reaction of small-scale vorticity on the straining field

A standard working hypothesis for three-dimensional turbulence is that small spatial scales react on larger scales like an effective eddy viscosity, augmenting molecular viscosity. Some supporting analytical evidence is provided by closure approximations of the direct-interaction family. These closures yield a rate of loss of energy from the larger scales that is proportional to the mean-square rate of strain associated with the larger scales and to an eddy viscosity which depends explicitly only on the small scales. The eddy viscosity is proportional to the total energy in the small scales (per unit mass) and to a characteristic dynamical time of the latter. The closure approximations yield this picture exactly in the limit of very great scale separation and approximately for more moderate separations (Kraichnan 1966).

The eddy-viscosity hypothesis clearly requires drastic revision for two-dimensional turbulence, since the asymptotic enstrophy-transferring inertial range (if it exists) involves a zero rate of energy transfer to smaller scales (higher wavenumbers). Consider a wavenumber k which lies within such an inertial range. Let $T_{>k}(p)$ be the rate at which energy is transferred into a unit wavenumber interval at $p < k$ owing to interactions with all wavenumbers $> k$. Positive overall enstrophy transfer and zero overall energy transfer give the simultaneous conditions

$$\int_0^k T_{>k}(p)\,dp = 0, \quad \int_0^k T_{>k}(p)\,p^2 dp < 0. \tag{5.1}$$

$T_{>k}(p)$ cannot be negative for all $p < k$, as would be anticipated by analogy with three-dimensional turbulence. If $T_{>k}(p)$ is negative for p near the boundary at $p = k$, then, to satisfy (5.1), it must be positive for still smaller p. This suggests that wavenumbers $\gtrsim k$ may exert a *negative* eddy viscosity on wavenumbers $p \ll k$.

A negative eddy viscosity is further suggested by conservation requirements in the straining of small scales by much larger scales. We expect that the straining tends to stretch the small scales into thin structures thereby increasing their characteristic wavenumbers k. But since vorticity must be conserved in the straining, and the ratio of energy to enstrophy is k^{-2}, the straining implies a loss of energy by the small scales. By energy conservation, the lost energy must appear in the straining scales. Conservation could be satisfied if there were an eddy viscosity proportional, as in three dimensions, to the mean-square rate of strain in the large scales, the energy in the small scales and the characteristic dynamical time of the latter, but negative in sign. If the dynamics of the small scales are dominated by the straining, this characteristic dynamical time should be the reciprocal of the rate of strain itself.

Further consideration, however, suggests that the numerical coefficient in the asymptotic eddy viscosity expression for $p/k \to 0$ may, in fact, vanish. This is because of a peculiarity of random straining that is unique to two dimensions. Returning to the passive-scalar example, consider the straining of little blobs of scalar by a large-scale, isotropic, statistically stationary straining field which is externally forced so as to vary very rapidly in time. Let the blobs have initially a many-period sinusoidal amplitude profile such that their initial spectrum peaks sharply at a wavenumber k_i. Then the probability distribution of k, the wavenumber of a blob at time t, is lognormal and the moments are

$$\langle (k/k_i)^n \rangle = \exp[(n+n^2D^{-1})\bar{A}t] \tag{5.2}$$

(Kraichnan 1974*b*). Here D is the dimensionality and

$$\bar{A} = (D+2)^{-1}\int_{-\infty}^{t} \langle [\partial u_i(\mathbf{x},t)/\partial x_j][\partial u_i(\mathbf{x},s)/\partial x_j]\rangle\, ds, \tag{5.3}$$

where u_i is the straining velocity.

The right-hand side of (5.2) grows with t if $n > 0$. This is evidence that higher wavenumbers are produced. But since the straining is random, it also produces smaller wavenumbers with some probability, and the right-hand side of (5.2) also grows with t if n is sufficiently negative. Now consider $n = -2$. For $D = 3$, the right-hand side of (5.2) shrinks with increasing t. But for $D = 2$, it is independent of t. If the blobs were passively strained vorticity, then $\langle (k/k_i)^{-2}\rangle$ would measure the mean kinetic energy associated with the vorticity field. Thus, because the flow is two-dimensional, rapid straining like that just described would not actually decrease the kinetic energy. There is just enough back transfer of enstrophy to lower wavenumbers to balance the kinetic-energy reduction associated with transfer into higher wavenumbers. This implies that the asymptotic negative eddy-viscosity coefficient for $p/k \to 0$ vanishes.

Of course, it cannot be asserted from the above argument that the kinetic energy of a strained small-scale vorticity field also remains constant in the more realistic case of a straining field with finite correlation time, typically of the order of the eddy circulation time or, equivalently, the reciprocal rate of strain. According to the approximate closures of the direct-interaction family, the kinetic energy does stay constant in this case also, a result which will be presented in another paper. But even if the negative eddy viscosity is non-zero, the reaction on the straining field should be negligible if the total kinetic energy of this field is large compared with that of the strained vorticity field. To see this, let v_p and p be the typical velocity and wavenumber of the straining field while v_k and k are the corresponding quantities for the strained field. The typical eddy circulation time for the straining field is then $(v_p p)^{-1}$ and the typical rate of strain is $v_p p$. If the straining does produce a non-zero loss of kinetic energy from the strained small scales, its rate per unit mass must have the magnitude $(v_p p) v_k^2$. This represents a gain per eddy circulation time for the straining field of v_k^2. Thus the fractional gain is v_k^2/v_p^2, which goes to zero with v_k/v_p.

6. Intermittency effects in the enstrophy cascade

We wish now to discuss the build-up and consequences of possible intermittency in the enstrophy cascade under the assumption, supported in §5, that the reaction of strained vorticity fluctuations on a straining field of much smaller wavenumber and much higher kinetic energy is negligible. Consider first a special case, for which the analogy with convection of a passive scalar seems unassailable. Let the gravest modes k_0 be excited to a typical velocity v_0 by some kind of external forcing which makes the velocity field statistically stationary with correlation time $\lesssim (v_0 k_0)^{-1}$. (This could be accomplished, for example, by adding to the Navier–Stokes equation both a random forcing term and an extra damping term for these modes.) Because both energy and enstrophy are conserved by the nonlinear terms, this velocity field cannot escape to higher wavenumbers. Now let additional forcing be added at a wavenumber $k_1 \gg k_0$ so that vorticity is pumped in, statistically isotropically and homogeneously, and at a statistically stationary rate small enough that the total kinetic energy pumped in at k_1 remains $\ll v_0^2$ for a time $\gg (v_0 k_0)^{-1}$, and, moreover, the total enstrophy pumped in remains $\ll v_0^2 k_0^2$ for this time.

There is then a perfectly clean separation between strained and straining fields, negligible reaction on the straining field and nothing to flaw the correspondence with passive-scalar convection. Consequently, the same intermittency effects as these inferred for the passive-scalar case (Kraichnan 1974*b*) must also arise for the enstrophy cascade to wavenumbers $k > k_1$. These effects were found exactly for the case of a rapidly varying straining field (*loc. cit.* §§3, 4 and 6). The results imply, first, that a region of k^{-1} enstrophy spectrum (k^{-3} energy spectrum) develops for $k > k_1$ and that the top wavenumber of this range increases exponentially with t until wavenumbers are reached where viscous dissipation prevents further growth. If $\nu = 0$, the range grows until $k_{\max}$ is reached. At times before the viscous cut-off is reached, the intermittency of spatial derivatives of the enstrophy field grows rapidly with t. The growth is exponential, as measured by kurtoses, and the rate of growth increases with the order of the derivative. Contributions to these derivatives come principally from the roll-off region at the top of the growing k^{-1} range. Intermittency in the k^{-1} range proper, as measured by the statistics of vorticity differences across distances equal to the reciprocal of a wavenumber in the range, increases very slowly with growth of the range.

In the steady-state k^{-1} regime achieved after viscous effects are fully felt, intermittencies of the spatial derivatives of vorticity are much weaker, and grow less rapidly with the order of the derivative, than they would be in an inviscid k^{-1} regime of the same extent in wavenumber. This is because the excitation falls much more sharply with increasing k in the dissipation range than it does in the roll-off region at the top of the inviscid k^{-1} range.

All the above results, although obtained in exact analytical form only in the case of a very rapidly varying straining field, were found in the scalar investigation to be qualitatively valid also in the case of a straining field whose correlation time is of the order of its eddy circulation time, provided that the time of evolution is long compared with this correlation time. If the correlation time of the straining field is of the order of the reciprocal rate of strain, then, apart from viscous effects at high enough wavenumbers, the typical rate of strain gives the exponential growth rates for the top wavenumber of the k^{-1} range and for the build-up of intermittencies, to within numerical factors. For a straining field forced to vary very rapidly in time, these growth rates are instead given by the quantity $\bar{A}$ defined in (5.3).

It should be emphasized that the univariate distribution of the vorticity field, in contrast to the univariate distributions of the spatial derivatives, does not become more intermittent as a result of the straining process, since the vorticity of each fluid element is an inviscid constant of motion. Changes in the univariate vorticity distribution can arise only from viscous effects.

The intermittency build-up has no effect on the k^{-1} spectrum law itself because the dynamics of the strained vorticity field are linear. The linearity implies that the equations for evolution of the spectrum involve only second-order moments of the vorticity field and decouple from those for higher-order moments. There is also no correction to the k^{-1} law of the logarithmic kind displayed in (1.3). This is because our assumption of very weak excitation except at the lowest wavenumbers implies that the total contribution of the k^{-1} range to the straining field is negligible. The absence of this correction is again associated with the linearity of the dynamics.

Now consider the more physically relevant case in which there is no forced excitation at k_0. Instead, take $k_0 \ll k_1 \ll k_{\max}$ and let energy be fed in at k_1 by an isotropic, statistically steady forcing, starting from an initial state of zero excitation at all wavenumbers. In this case, the arguments of Kraichnan (1971) and Leith & Kraichnan (1972) suggest that the system seeks equilibrium by a transfer of kinetic energy to $k > k_1$ and of enstrophy to $k > k_1$, the latter transfer proceeding by way of an inertial range of the form (1.3). This case differs from the one just considered in that the inertial range itself provides the straining field. The dynamics of straining in this range are then not linear, and this is what leads to the logarithmic correction in (1.3). We wish now to examine the self-consistency of (1.3), and the associated dynamical picture, under intermittency effects.

We shall start by recalling the physical reasoning which leads to the logarithmic correction. If the asymptotic spectrum had exactly the form (1.2), then the total enstrophy and, by inference, the mean-square rate of strain would diverge logarithmically with increases in the top wavenumber of the range. If k is a wavenumber well within the range, then cancellation effects over distances $\sim k^{-1}$ should reduce the effectiveness of strain-field components of wavenumbers $> k$ in distorting structures of wavenumber k, with the result that the contribution of wavenumbers $> k$ to enstrophy transfer at k converges even though the total mean-square rate of strain does not. However, we are left with an effective mean-square rate of strain, acting at k, which increases like $\ln(k/k_1)$. The k^{-3} range then cannot survive because enstrophy transfer increases with k. The corrected equilibrium, in which transfer is independent of k, has the more steeply falling form (1.3). The mean-square enstrophy in wavenumbers $< k$, and hence the effective rate of strain acting at k, is still a logarithmically increasing function of k/k_1, although a weaker one.

If k/k_1 is very large the effective strain acting at k will be dominated by wavenumbers $\ll k$, since such wavenumbers will comprise most of the logarithmic extent of the inertial range below k. Hence the straining dynamics resemble those of the previous case, with the straining field concentrated at k_0, in the sense that they are non-local. We can again argue, although now not so unassailably, that the reaction of the strained structures of wavenumber k on the straining field is asymptotically negligible for typical k in a very long inertial range. Moreover, the effective straining field should exhibit substantial statistical independence among the various decades in wavenumber which contribute to it because each decade is connected by the constant-in-the-mean transfer rate to a different time of random input at k_1.

We have, then, all the ingredients for the build-up of intermittency effects qualitatively similar to those for the case of a random straining field concentrated at k_0. The key feature is that the dynamics still exhibit an effective linearity because the strained flow structures of any given scale size in the inertial range react negligibly on an effectively random straining field. Since the predicted intermittency is for vorticity derivatives and not for the vorticity field itself, an increase of intermittency with wavenumber should not substantially affect the statistics of the effective straining field and hence should not upset the spectrum law (1.3). While the increase of intermittency parameters with the length of the range (1.3) should be qualitatively similar to that for a scalar field in the k^{-1} range, these parameters should be even closer to those of a passive scalar strained by a random velocity field with spectrum (1.3).

Now suppose that there is initial random excitation at $k = k_1$ but that the system evolves freely towards equilibrium thereafter, with no forcing. There appears to be no obvious inconsistency in the internal dynamics behaving as in the forced case. If so, there should be a similar growth of a log-corrected k^{-1} range and a similar increase of intermittency with time, up to the time when the k^{-1} range extends either to k_{max} or to wavenumbers where viscosity is significant. The principal differences should be connected with the fact that the enstrophy in the present case is an inviscid constant of motion, instead of increasingly linearly with time. This implies that the effective rate of strain felt by a given wavenumber k in the k^{-1} range should decrease with time in proportion to the logarithmic fraction of the range that lies below k. Both $E(k)$ and η in (1.3) should decrease with time.

The arguments seem less compelling in this freely evolving case, however, because, in the absence of random forcing, there would appear to be more opportunity for correlations to build up between widely separated scales. We have no answer to this, except to say that it is unclear what form the correlations could take to spoil our conclusions. We have noted in §4 that the vorticity cannot concentrate in small regions and so escape a straining field. This is because the vorticity of each fluid element is a constant of motion, and the fluid is incompressible. It therefore seems difficult to escape the straining of vorticity into smaller scales. Once vorticity enters these scales, its associated velocity field is bounded by the k^{-2} relation between kinetic energy and enstrophy, and it becomes hard to see how the small scales can react significantly upon the effective straining field they experience.

Suppose now that viscosity is zero and the behaviour argued above for the freely evolving system actually does occur. The growth of the k^{-1} range would continue until the excitation reached $k_{\max}$. There would then presumably be a gradual approach towards the absolute equilibrium state in which the enstrophy is in equipartition. It seems plausible that the self-straining would continue to pump enstrophy towards $k_{\max}$ until the equipartition level of excitation was reached at $k \sim k_{\max}$ and that then the equipartition state would work down to lower wavenumbers. If all this is so, we have the paradox of an eventual Gaussian equilibrium state being reached through a transient phase in which intermittencies increase and reach strongly non-Gaussian levels if $k_{\max}$ is large enough. In the truncated system, constancy of vorticity in each fluid element breaks down when wavenumbers $\sim k_{\max}$ are reached. The straining picture from which we inferred the increase of intermittency no longer is valid then, and there is the opportunity for relaxation of the intermittency and approach to the Gaussian statistics.

7. The downward energy cascade

The similarity state proposed by Batchelor (1969), Leith (1968) and Kraichnan (1967) involves a cascade of energy to wavenumbers smaller than the input wavenumber k_1 through a backward-transferring Kolmogorov range of the form (1.1), as well as the upward enstrophy cascade. In contrast to the enstrophy range, the dynamics of the $k^{-\frac{5}{3}}$ range, if it does occur, are local in the sense that most of the effective strain acting at a given wavenumber in the range arises from wavenumbers of the same magnitude. The self-consistency arguments for this are the same in two as in three dimensions (Kraichnan 1967). It is then *a priori* plausible that intermittency effects can arise like those which probably affect the $k^{-\frac{5}{3}}$ range in three dimensions (Kraichnan 1974*a*). In the three-dimensional case, these intermittency effects increase the efficiency of cascade with each nominal cascade step. Since the rate of energy flow is constant through the range, this means that $E(k)$ falls off more rapidly towards high k than (1.1). If intermittency in the two-dimensional cascade gives a similar increase of efficiency with the number of cascade steps, then $E(k)$ should fall below (1.1) towards lower k. That is, the negative exponent in the power law for $E(k)$ should have an absolute value *smaller* than $\frac{5}{3}$.

If intermittency does build up in the downward cascade of energy, this transient, non-equilibrium phenomenon seems a more plausible explanation than the structure of the absolute equilibrium ensembles for the computer experiments reported by Joyce & Montgomery (1973) and Edwards & Taylor (1974). These experiments show clumping of like-signed vorticity on scales larger than the input scale and smaller than the box size. We have noted that the conservation laws prohibit more than a small fraction of the initial vorticity from entering such clumps. But this does not outlaw them altogether, nor prevent them from representing a substantial fraction of the energy, in contrast to the vorticity. It may be that such clumps are one physical expression of the tendency towards intermittency in the downward cascade that leads asymptotically to a modified $-\frac{5}{3}$ law. On the other hand, the negative-temperature absolute equilibrium states involve an enhanced occupancy of the ground-state modes k_0. This means a smooth distribution of velocity and would not appear to provide an explanation for the irregular clumping observed in the experiments.

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